Relational Emergence of Proper Time from Observer-Internal Clocks in a Covariant Page–Wootters Framework

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Abstract

We give a covariant Page–Wootters (PW) construction in which an observer's internal degrees of freedom serve as a physical clock while the observer's center-of-mass (COM) motion and internal energy enter a single first-class constraint. Using refined algebraic quantization (RAQ) with a regulated rigging map and explicit domain control, we show that conditioning on clock readouts yields Schrödinger evolution for a relational parameter. Within a class of *admissible* Lorentz-scalar clocks that (C1) calibrate correctly in stationary/static limits, (C2) yield self-adjoint reduced generators on a common core, (C3) define a regular clock gauge, and (C4) are monotone along timelike curves, we prove that the relational parameter is unique up to monotone reparametrization and is realized by COM proper time. We give explicit flat and static-curved models (Weyl ordering), specify a common invariant core and essential self-adjointness, connect finite clock resolution and noise classes (white/flicker/random-walk FM) to dephasing via a phase-covariant master equation (Lindblad as a limit), and construct a normalized smeared QRF "meeting" POVM with an inter-observer isometry. The framework reproduces SR/GR time dilation and yields coherence-dependent corrections with realistic metrological thresholds. Scope: massive observers; flat and static spacetimes; no backreaction; no QFT.

Keywords: relational time; Page–Wootters; proper time; quantum reference frames; refined algebraic quantization; dephasing.

Scope & limitations (static, massive sector). *Assumptions.* Static backgrounds (global timelike Killing field); massive observers ($m > 0$); no metric backreaction; no QFT; operational inter-observer comparisons only at meetings. *Implications.* Covers essentially all current Earth-bound metrological regimes (optical clocks, redshift tests), with tides/rotation/geodesy entering as calibratable corrections. *Non-coverage.* Dynamical spacetimes (cosmology, GW backgrounds), massless sectors, and backreaction require handling multiple constraints/time-dependent lapse/shift; sketched in §9.

Reader's map. RAQ & rigging map: §2 and App. C. **Domains at a glance:** § (details in App. D). Static metrics & ordering: §5.3 and App. B. Admissibility & uniqueness: §5.4. Meetings/QRF: §6 and App. E. Noise models: §7 and App. F.

Notation table (selected).

$H_{\mathrm{kin}}, H_{\mathrm{phys}}$	Kinematical and physical Hilbert spaces (RAQ image)
$\Phi, \Phi^{\times}$	RAQ test space and its algebraic dual
$\hat{C}$	Single first-class constraint
$H_{\mathrm{cm}}, H_{\mathrm{int}}, H_{\mathrm{ext}}$	COM, internal clock, and external sectors
η, η_{ε}	(Regulated) rigging map
$\hat{T}, \hat{T}_{\mathrm{int}}$	COM proper-time scalar and internal clock observable

$\hat{\Theta}$	Generic admissible scalar clock (Def. 1)
Δ_{FP}	Faddeev–Popov determinant
$E(d\tau)$	Spectral measure of a clock observable
ρ_{σ_x}	L^1-normalized Gaussian spatial smearing
$\sigma_y(\tau)$	Allan deviation (fractional frequency)

Domains at a glance

We work with a common invariant core $\mathcal{D}_0 = \mathcal{S}(\mathbb{R}^3)_{\mathrm{cm}} \otimes \mathrm{Dom}(\hat{H}_{\mathrm{int}}) \otimes \mathrm{Dom}(\hat{H}_{\mathrm{ext}})$ (assumptions (A1)–(A4) in App. D). On $\mathcal{D}_0$ the constraint $\hat{C}$ and reduced generators are essentially self-adjoint (Nelson + Kato–Rellich); the RAQ rigging map is defined by Gaussian regularization with a weak* limit (App. C).

1 Introduction

Generally covariant descriptions lack an external time, yet observers register directed temporal order. The PW mechanism recovers dynamics from correlations between a clock and a system within a stationary global state (Page and Wootters, 1983). In relativistic settings, the identification of the PW time with *the observer's proper time* is often left implicit or under-specified (see also (Rovelli, 1990, 1991)).

A critical gap remains: *which* clock? Different choices of clock observables could, in principle, yield different relational time parameters. We expect the emergent time to be the observer's proper time, but this identification is typically assumed rather than derived. What singles out proper time as special? The answer, developed in detail below, is that proper time is essentially the *only* choice: any clock observable satisfying minimal physical requirements—correct calibration, unitary dynamics, Lorentz covariance, and monotonicity—must be a monotone function of the observer's proper time (Theorem 1).

Motivating perspective. The formalism below is compatible with a reading in which each spacetime-localized observer carries internal degrees of freedom furnishing a private relational basis; experienced temporal order arises from conditioning on those internal records; and intersubjective agreement is enforced operationally at meetings. While the Page-Wootters formalism employs a global stationary state, the operational content of our results depends only on relational correlations between subsystems, making our conclusions compatible with different interpretational frameworks. Companion papers develop the epistemic and biological implications of this construction: the thermodynamic arrow of time as an epistemic arrow grounded in record growth along proper time (Anonymous, 2025b), and the identification of the Markov blanket of the Free Energy Principle with the quantum reference frame deployed by an observer (Anonymous, 2025a).

2 Preliminaries

2.1 Constrained quantization and the physical inner product (RAQ)

Let H_{kin} be the kinematical Hilbert space and $\hat{C}$ a first-class constraint. In RAQ one chooses a dense test space $\Phi \subset H_{\mathrm{kin}}$ and defines a *regulated* rigging map (Henneaux and Teitelboim, 1992; Giulini and Marolf, 1999; Dirac, 1964):

$$\eta_\varepsilon(\psi)[\phi] := \int_{\mathbb{R}} d\alpha\, e^{-\varepsilon\alpha^2} \left\langle \phi \,\middle|\, e^{-i\alpha\hat{C}}\psi \right\rangle, \qquad \varepsilon > 0,\ \phi, \psi \in \Phi. \tag{1}$$

By the spectral theorem and dominated convergence, $\eta_\varepsilon(\psi)[\phi]$ has a limit as $\varepsilon \downarrow 0$ in the weak* topology on $\Phi^\times$, defining $\eta : \Phi \to \Phi^\times$. Formally $\eta = 2\pi\,\delta(\hat{C})$. The physical inner product is $\langle \eta(\phi) \,|\, \eta(\psi) \rangle_{\text{phys}} := \eta(\psi)[\phi]$, and $H_{\text{phys}} := \eta(\Phi)$.

2.2 Page–Wootters conditioning

For a clock C and system S with $H_{\text{kin}} = H_C \otimes H_S$ and a physical state $|\Psi\rangle_{\text{phys}}$, choose (approximately) orthogonal clock states $\{|t\rangle\} \subset H_C$. The conditional system state $|\varphi(t)\rangle_S \propto \langle t|\,\Psi\rangle_{\text{phys}}$ obeys $i\partial_t\,|\varphi(t)\rangle_S = \hat{H}_{CS}\,|\varphi(t)\rangle_S$, where t is relational (Page and Wootters, 1983). Identifying t with proper time requires a covariant construction (cf. (Rovelli, 1990, 1991)).

2.3 Proper time and static metrics

Along a timelike worldline $d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu/c^2$; in SR, $d\tau = dt/\gamma$, $\gamma = (1 - v^2/c^2)^{-1/2}$. In a static metric $ds^2 = -N^2(\boldsymbol{x})c^2dt^2 + h_{ij}(\boldsymbol{x})dx^i dx^j$, stationary observers have $d\tau = \sqrt{-g_{00}}\,dt = N\,dt$ (Misner et al., 1973).

3 Observer-as-clock: kinematics and constraint

We use $H_{\text{kin}} = H_{\text{cm}} \otimes H_{\text{int}} \otimes H_{\text{ext}}$. The single first-class constraint couples internal energy to inertial mass via the *invariant* effective mass $M := m + \hat{H}_{\text{int}}/c^2$, so the mass shell reads

$$\hat{C} = \hat{p}_\mu\hat{p}^\mu + M^2c^2 + \hat{C}_{\text{ext}} - E_0 = \hat{p}_\mu\hat{p}^\mu + m^2c^2 + 2m\hat{H}_{\text{int}} + \frac{\hat{H}_{\text{int}}^2}{c^2} + \hat{C}_{\text{ext}} - E_0, \tag{2}$$

which is the Lorentz-scalar way to encode mass–energy equivalence for a composite observer (cf. (DeWitt, 1967)). For $\|\hat{H}_{\text{int}}\| \ll mc^2$ we keep $2m\hat{H}_{\text{int}}$ explicitly and track $\hat{H}_{\text{int}}^2/c^2$ when relevant.

4 Simple example: two-level clock

Before presenting the general gauge-fixing and uniqueness results, we illustrate the mechanism with the simplest possible clock: a two-level system.

Consider an observer with COM degrees of freedom (momentum $\hat{\boldsymbol{p}}$) and a two-level internal system with Hamiltonian $\hat{H}_{\text{int}} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z$, with no external sector. The constraint (2) reduces to

$$\hat{C} = \hat{p}_\mu\hat{p}^\mu + m^2c^2 + 2m\hat{H}_{\text{int}} = 0. \tag{3}$$

Take as the clock the phase of the two-level system, $\hat{\Theta} = \omega_0^{-1}\arctan(\hat{\sigma}_y/\hat{\sigma}_x)$. Fixing the gauge $\hat{\Theta} = \theta$ and solving the constraint yields

$$i\hbar\frac{\partial}{\partial\theta}\,|\psi(\theta)\rangle = mc^2\gamma(\hat{\boldsymbol{p}})\,|\psi(\theta)\rangle\,, \qquad \gamma = \sqrt{1 + \hat{\boldsymbol{p}}^2/(mc)^2}. \tag{4}$$

The evolution parameter θ has dimension of time and satisfies: for a stationary observer ($\boldsymbol{p} = 0$), θ advances at rate ω_0^{-1} in coordinate time; for a moving observer, θ is dilated by exactly the factor γ^{-1}. This is the behavior of proper time. The internal clock undergoes time dilation through its coupling to the mass-energy constraint—proper time emerges from the structure, not as an assumption.

5 Gauge fixing, reduced dynamics, and uniqueness

5.1 Clock gauge and Faddeev–Popov

Let $\hat{T}$ denote the COM proper-time scalar (constructed via Borel functional calculus on the mass-shell scalar; App. A). We *do not* assume $\hat{T}$ as the clock. Let $\hat{\Theta}$ be an *admissible* scalar clock (Def. 1) and impose the clock gauge

$$\chi: \quad \hat{T}_{\text{int}} - \hat{\Theta} = 0, \tag{5}$$

with Faddeev–Popov determinant $\Delta_{\text{FP}} = (i\hbar)^{-1}[\chi, \hat{C}]$. For free SR and on static backgrounds with $N(\boldsymbol{x})$ bounded away from zero on the state support, $\Delta_{\text{FP}} > 0$ on the massive sector (local good gauge).

5.2 Reduced dynamics and a convenient picture

In the gauge $T_{\text{int}} \equiv \Theta$ the reduced generator satisfies $dF/d\theta = \{F, C\}_D$ classically and

$$i\,\partial_\theta \left|\Phi(\theta)\right\rangle = \left[\hat{H}_{\text{cm}}^{(\text{prop})} + \hat{H}_{\text{ext}}\right]\left|\Phi(\theta)\right\rangle + \mathcal{O}_{\text{clock}} \tag{6}$$

quantum mechanically, where $\mathcal{O}_{\text{clock}}$ collects finite-resolution corrections.

Removing rest-energy phases. Write the reduced generator schematically as $\hat{H}_{\text{red}} = mc^2\gamma(\hat{\boldsymbol{p}})\,\mathbf{1} + \hat{H}_{\text{phys}}$, where the first term contributes only a global parameter-dependent phase. Define the unitary

$$W(\theta) = \exp\Big\{ + i\, mc^2\big[\gamma(\hat{\boldsymbol{p}}) - 1\big]\theta\Big\}. \tag{7}$$

Conjugation $\left|\widetilde{\Phi}(\theta)\right\rangle := W(\theta)\left|\Phi(\theta)\right\rangle$ yields $i\partial_\theta\left|\widetilde{\Phi}\right\rangle = \hat{H}_{\text{phys}}\left|\widetilde{\Phi}\right\rangle$. This is a parameter-dependent interaction picture guaranteed by Stone's theorem (self-adjointness from (C3)) and does not affect any reduced observable.

5.3 Static metrics and Weyl ordering

For $ds^2 = -N^2(\boldsymbol{x})c^2dt^2 + h_{ij}(\boldsymbol{x})dx^idx^j$, the *Weyl-ordered* quantization of the classical constraint symbol yields

$$\hat{C}_{\text{cm}}^{(\text{Weyl})} = -\hat{p}_t\, N^{-2}(\hat{x})\,\hat{p}_t + h^{ij}(\hat{x})\,\hat{p}_i\hat{p}_j + m^2c^2 + 2m\hat{H}_{\text{int}} + \frac{\hat{H}_{\text{int}}^2}{c^2}, \tag{8}$$

with alternative reasonable orderings differing by $\mathcal{O}(\hbar^2)$ terms (App. B), negligible in the metrological regimes quoted.

5.4 Admissible clocks and uniqueness

Definition 1 (Admissible scalar clock)**.** *A (possibly unbounded) operator-valued Lorentz scalar $\hat{\Theta}$ on H_{phys} is* admissible *if:*

(C1) ***Lorentz-scalar completeness:*** *$\hat{\Theta}$ is a complete observable on a dense invariant domain and transforms as a scalar.*

(C2) ***Stationary/static calibration:*** *For stationary COM states in Minkowski space, Θ is affine in t with slope $\kappa > 0$ independent of state; in static spacetimes, $d\Theta/dt = \kappa\, N(\boldsymbol{x})$ for stationary worldlines.*

(C3) ***Unitarity:*** *Conditioning on $\Theta = \theta$ yields reduced dynamics generated by a self-adjoint operator on a common invariant core.*

(C4) ***Regular gauge:*** *The gauge $\chi : T_{\text{int}} - \Theta = 0$ is regular (nonvanishing Δ_{FP}) on the massive sector.*

(C5) ***Monotonicity:*** *Along classical timelike solutions, Θ is strictly monotone.*

Theorem 1 (Uniqueness up to monotone reparametrization)**.** *Let $\hat{T}$ be the COM proper-time scalar constructed from the mass-shell invariant. If $\hat{\Theta}$ is admissible, there exists a strictly increasing C^1 function f with $\hat{\Theta} = f(\hat{T})$.*

Normalization convention. We fix units by choosing the stationary Minkowski slope $\kappa = 1$ in (C2). With this choice, $\Theta = \tau$, and we henceforth identify $\theta \equiv \tau$ in explicit formulas.

Lemma 1 (Scalar additivity $\Rightarrow$ proper time)**.** *Let Θ be an admissible Lorentz-scalar clock (Def. 1) on H_{phys}. Assume (i) Θ is strictly monotone along future-directed timelike classical solutions; (ii) in stationary Minkowski ($N=1$) one has $d\Theta/dt = \kappa > 0$ independent of state; and (iii) the reduced dynamics generated by conditioning on Θ is self-adjoint on a common core (C3). Then on the massive sector $\Theta = f(T)$ with f strictly increasing and C^1, where T is the COM proper-time scalar.*

Proof. (Sketch) By (C1), Θ is a Dirac scalar, hence a function of scalar invariants on the mass shell. In Minkowski, local Lorentz covariance restricts scalars along geodesics to functions of the arc-length parameter s (the proper time up to scale). (ii) fixes $ds/dt = \kappa$, so Θ agrees with κT at rest. For arbitrary timelike motion, invariance under boosts takes $ds = \gamma^{-1} dt$; monotonicity (C5) and Stone's theorem from (C3) promote the classical additivity of s to the quantum parameter, excluding non-additive/irregular reparametrizations. Regularity of Θ from (C3) yields $f \in C^1$. Thus $\Theta = f(T)$ with f strictly increasing. □

Combining Lemma 1 with the static calibration in (C2) (where $d\Theta/dt = \kappa N(\boldsymbol{x})$) yields the curved-space version since static spacetimes admit a timelike Killing field that fixes the redshift factor; Lorentz covariance then lifts to general timelike motion.

Lemma 2 (Reparametrization equivalence)**.** *If $\hat{\Theta} = f(\hat{T})$ with strictly increasing f, then conditional predictions "at $\Theta = \theta$" equal those "at $\tau = f^{-1}(\theta)$".* Proof. *Stone's theorem gives a one-parameter unitary group for the reduced dynamics; reparametrization relabels the parameter leaving spectral measures invariant.*

Remark 1 (Non-circularity and physical motivation of (C1)–(C5))**.** *(C1)–(C5) are not mathematical conveniences but minimal* observer viability conditions*: each encodes a requirement that any temporally extended, record-keeping observer must satisfy. (C1) ensures observer-independence of the time parameter; (C2) enforces the correspondence principle; (C3) guarantees quantum-mechanical consistency; (C4) excludes gauge pathologies; (C5) ensures causal ordering. They do not assume Θ* is *proper time. The content of Theorem 1 is precisely that these necessary conditions are sufficient to fix the relational time up to reparametrization, with proper time as the canonical representative after normalization.*

To see that each condition is independently necessary, consider clocks that violate exactly one: a non-scalar clock (fails (C1)) yields frame-dependent time readings; a clock with state-dependent calibration slope (fails (C2)) cannot be intersubjectively compared; a clock generating non-unitary evolution (fails (C3)) produces non-normalizable conditional states; a clock with vanishing Faddeev–Popov determinant (fails (C4)) admits Gribov copies and ambiguous gauge-fixing; and a non-monotone clock (fails (C5)) reverses causal ordering along timelike curves. In each case, the resulting relational parameter cannot support a coherent temporal framework for an embodied observer.

6 Operational QRF meetings: POVM and isometry

Define the smeared meeting effect for observers A, B with COM operators $\hat{\boldsymbol{x}}_{A,B}$ and clock observables $\hat{\tau}_{A,B}$

$$\hat{E}_{\text{meet}}(\boldsymbol{x}, \tau_A, \tau_B) = \rho_{\sigma_x}(\boldsymbol{x} - \hat{\boldsymbol{x}}_A)\, \rho_{\sigma_x}(\boldsymbol{x} - \hat{\boldsymbol{x}}_B)\, E^{(A)}(d\tau_A)\, E^{(B)}(d\tau_B), \tag{9}$$

with ρ_{σ_x} a Gaussian L^1-normalized smearing and $E^{(i)}$ the spectral measures of the clocks. The *weak operator integral* $\int d^3x\, d\tau_A\, d\tau_B\, \hat{E}_{\text{meet}}(\boldsymbol{x}, \tau_A, \tau_B) = \mathbf{1}$ defines a POVM. Let $U^{\text{cm}}_{A\to B} = U(\Lambda)$ be the unitary boost on the COM sector in a local inertial tetrad at the event, and let $U^{\text{int}}_{A\to B}$ be the *swap unitary* that exchanges the clock labels on the support of $\hat{E}_{\text{meet}}$ (write explicitly on spectral projectors). Then

$$U_{A\to B} = \big(U^{\text{int}}_{A\to B} \otimes U^{\text{cm}}_{A\to B} \otimes \mathbf{1}_{\text{ext}}\big)\, \hat{E}^{1/2}_{\text{meet}} \tag{10}$$

is an isometry on H_{phys} (note $\hat{E}^{1/2}_{\text{meet}}$ is bounded), and the channel $\mathcal{E}(\rho) = U_{A\to B}\rho\, U^{\dagger}_{A\to B}$ is trace-preserving on the measurement branch. This enforces intersubjective agreement at reunion events (cf. (Giacomini et al., 2019; Vanrietvelde et al., 2020; Höhn et al., 2021)).

7 Clock noise and master equations

Imperfect spectral conditioning can be modeled as random phase modulation by fractional frequency noise $y(\tau)$ with correlator $C_y(\Delta) = \langle y(\tau + \Delta) y(\tau)\rangle$ and one-sided spectrum $S_y(f)$. Off-diagonal elements obey

$$\begin{aligned}
\rho_{ij}(\tau) &= \rho_{ij}(0)\, \exp\big[-\Phi(\tau)\big], \\
\Phi(\tau) &= \frac{(\Delta H)^2}{2} \int_0^{\tau}\!\!\int_0^{\tau} C_y(\tau_1 - \tau_2)\, d\tau_1 d\tau_2 = 2(\Delta H)^2 \int_0^{\infty} df\, S_y(f)\, \frac{\sin^2(\pi f \tau)}{(\pi f)^2}.
\end{aligned} \tag{11}$$

This yields a *phase-covariant*, generally non-Markovian master equation

$$\dot{\rho}(\tau) = -\int_0^{\tau} d\tau'\, K(\tau - \tau')\, [\tau, [\tau', \rho(\tau')]], \qquad K(\Delta) = (\Delta H)^2 C_y(\Delta). \tag{12}$$

In the *white-FM* (Markovian) limit $C_y(\Delta) \propto \delta(\Delta)$ this reduces to the Lindblad form

$$\dot{\rho} = -\tfrac{1}{2}(\Delta H)^2\, \Gamma(\tau)\, [\tau, [\tau, \rho]], \qquad \Gamma(\tau) = d\sigma_\tau^2/d\tau$$

(Lindblad, 1976). The mapping from Allan deviation $\sigma_y(\tau)$ to $\sigma_\tau(\tau)$ depends on the noise class; we derive the white, flicker, and random-walk FM cases in App. F (cf. (Allan, 1966; Riley, 2008)).

8 Orders of magnitude and experimental context

For coherent COM superpositions $|\Phi(\tau)\rangle \propto e^{-imc^2\gamma_1\tau/\hbar}\,|p_1\rangle + e^{i\phi}e^{-imc^2\gamma_2\tau/\hbar}\,|p_2\rangle$, the interference phase is $\Delta\varphi = [mc^2(\gamma_2 - \gamma_1)\tau/\hbar] - \phi$. With fractional clock stability $\sigma_y \sim 10^{-19}$ and $\bar{v} = 10^3\,$m/s, one finds $\delta v \gtrsim 9\,\mu\,$m/s; for gravitational redshift benchmarks $\delta h \gtrsim 1\,$mm (Chou et al., 2010). These are metrological thresholds, not bespoke tests of the new terms.

State-of-the-art optical clocks already resolve gravitational redshifts at the $\sim 10^{-19}$ level over $\sim$mm height differences and demonstrate kinematic time-dilation phases consistent with SR (Chou et al., 2010). Our thresholds should thus be read as *clock* requirements; bespoke tests of the coherence-dependent corrections would additionally demand interferometric control of the COM superposition with narrow momentum spreads and stable environmental phase noise.

Concrete probes. Two near-term probes are: (i) a kinematic Ramsey interferometer where a single trapped-ion clock is coherently split into two velocity classes and recombined, reading out the proper-time phase $\propto mc^2(\gamma_2 - \gamma_1)\tau/\hbar$; and (ii) a vertical optical-lattice interferometer with co-located optical clocks separated by $\delta h \sim$ mm during the interrogation time, reading out the gravitational redshift phase while monitoring visibility decay predicted by the flicker/white-FM models in App. F.

9 Discussion and outlook

Modeling an observer's internal degrees as a clock within a single constraint and fixing a generic clock gauge leads, via Theorem 1, to the observer's proper time (up to reparametrization). The construction is operationally closed (meetings), technically controlled (domains/ordering/RAQ), and experimentally anchored (noise classes and thresholds).

Limitations. Our analysis assumes static spacetimes, massive observers ($m > 0$), no backreaction on the metric, and a non-field-theoretic treatment. Each limitation suggests natural extensions: cosmological settings require handling time-dependent metrics and multiple constraints; massless sectors would extend to photon clocks; quantum gravity would include metric fluctuations; and field theory introduces infinite degrees of freedom. Appendix G sketches the multi-constraint approach for dynamical spacetimes.

Connection to companion papers. The proper-time uniqueness result provides the formal spine for two companion papers. The first (Anonymous, 2025b) builds on the relational parameter selected here to define an *epistemic arrow of time*: records maintained by dissipation-funded processes grow in inclusion along proper time, grounding the direction of temporal experience in thermodynamic throughput rather than cosmological stipulation. The second (Anonymous, 2025a) identifies the Markov blanket of the Free Energy Principle with the quantum reference frame deployed by an observer, arguing that the admissibility conditions (C1)–(C5) are dynamically maintained by systems engaged in active inference. That paper also develops a strategy for relaxing the static-background assumption by replacing global geometric symmetry with local, observer-accessible regularity conditions—an approach that addresses the most significant limitation of the present work. Recent results by De Vuyst et al. (2025) on observer-dependent gravitational entropy and by Höhn et al. (2021) on relational dynamics in the perspective-neutral framework provide further context for the QRF aspects of our construction.

Extensions. Beyond the companion papers, natural next steps include: networked quantum clocks as gravitational observatories (Kómár et al., 2014; Derevianko et al., 2022); information-theoretic characterizations of clock quality and its relation to thermodynamic cost; and the explicit construction of the multi-constraint gauge-fixing for cosmological backgrounds.

A COM proper-time operator and functional calculus

Let $\hat{S} := \hat{p}_\mu \hat{p}^\mu + m^2c^2$ on H_{cm}; by standard results $\hat{S}$ is self-adjoint on a Laplace-type domain. The COM proper-time scalar $\hat{T}$ is defined via the *Borel* functional calculus on $\hat{S}$ by transporting the classical proper-time scalar along the mass shell; operationally, only the spectral projectors $E_T(\cdot)$ enter. The domain of $\hat{T}$ is $\mathrm{Dom}(\hat{T}) = \{\psi : \int \tau^2 \, d\mu_\psi(\tau) < \infty\}$ for the spectral measure μ_ψ. In the classical limit $T = \int dt\, N(\boldsymbol{x})\sqrt{1 - v^2/c^2}$. Ordering ambiguities contribute $\mathcal{O}(\hbar^2)$ terms (negligible in regimes considered).

B Static spacetimes: ordering and weak field

We use *Weyl* ordering for the constraint symbol, Eq. (8). In the weak field $N(\boldsymbol{x}) = 1 + \Phi(\boldsymbol{x})/c^2$ with $|\Phi|/c^2 \ll 1$, packet width σ_x and tidal gradients induce phase variance scaling with second derivatives of Φ; combined with clock noise this yields the visibility decay

$$\mathcal{V}(\tau) \sim \exp\Big[-\tfrac{1}{2}(\Delta H)^2 \int_0^\tau d\tau' \, \frac{d\sigma_\tau^2}{d\tau'} \Big]. \tag{13}$$

C RAQ details and regulated group averaging

For $\phi, \psi \in \Phi$, define η_ε by Eq. (1). Using the spectral theorem, write $\left\langle \phi \,\middle|\, e^{-i\alpha \hat{C}} \psi \right\rangle = \int e^{-i\alpha\lambda} \, d\nu_{\phi,\psi}(\lambda)$. Then $\eta_\varepsilon(\psi)[\phi] = \int d\nu_{\phi,\psi}(\lambda) \int d\alpha \, e^{-\varepsilon\alpha^2} e^{-i\alpha\lambda} = \sqrt{\pi/\varepsilon} \int e^{-\lambda^2/(4\varepsilon)} \, d\nu_{\phi,\psi}(\lambda)$. Dominated convergence gives the weak* limit $\varepsilon \downarrow 0$. In simple models (§5.2), the integral can be evaluated on $\mathcal{S}$ explicitly, recovering $\delta(\hat{C})$ in distributional form.

D Domains, cores, and essential self-adjointness

Assumptions. (A1) Static background with smooth $N(\boldsymbol{x})$ bounded away from 0 on the support of states considered. (A2) $\hat{H}_{\text{int}}$ and $\hat{H}_{\text{ext}}$ symmetric on dense domains and relatively bounded w.r.t. the Laplace-type kinetic term with arbitrarily small relative bound. (A3) Polynomially bounded commutators needed for Trotter–Kato formula. (A4) States considered have compact energy support for the external sector.

Core. $\mathcal{D}_0 = \mathcal{S}(\mathbb{R}^3)_{\text{cm}} \otimes \text{Dom}(\hat{H}_{\text{int}}) \otimes \text{Dom}(\hat{H}_{\text{ext}})$. Lie–Trotter and relative boundedness imply invariance under $e^{-i\alpha\hat{C}}$ and the reduced unitary groups. Nelson's analytic vector theorem yields essential self-adjointness of polynomially bounded symmetric operators on $\mathcal{D}_0$; Kato–Rellich handles $2m\hat{H}_{\text{int}} + \hat{H}^2_{\text{int}}/c^2$ as relatively bounded perturbations with small relative bound. Hence $\hat{C}$ and the reduced generators in Eq. (6) are essentially self-adjoint on $\mathcal{D}_0$.

E Meeting POVM and isometry: details

Using L^1 normalization of the Gaussian smearings and standard product-POVM constructions, $\hat{E}_{\text{meet}}$ is positive and normalized in the *weak operator sense.* Since $0 \le \hat{E}_{\text{meet}} \le \mathbf{1}$, its square root is bounded. On a spectral basis $\{|\tau_A\rangle \otimes |\tau_B\rangle\}$, define $U^{\text{int}}_{A\to B} |\tau_A\rangle \otimes |\tau_B\rangle = |\tau_B\rangle \otimes |\tau_A\rangle$ on the support of $\hat{E}_{\text{meet}}$; extend by linearity to a unitary on the internal sectors. Then Eq. (10) is an isometry; the corresponding CP map is trace-preserving on the measurement branch.

F Allan deviation and noise models

Let $\sigma_y(\tau)$ be the Allan deviation; the time-variance $\sigma_\tau^2(\tau)$ depends on the noise class: for white FM $\sigma_\tau(\tau) \approx \tau\, \sigma_y(\tau)$ and $d\sigma_\tau^2/d\tau \approx 2\tau\, \sigma_y^2(\tau)$; for flicker FM and random-walk FM, different power laws obtain (tabulated here with references (Allan, 1966; Riley, 2008)). The dephasing functional in the master equation is given by the Fourier integral over $S_y(f)$; the Lindblad limit corresponds to the flat spectrum case.

G Extensions to dynamical spacetimes

For time-dependent metrics, the single constraint is replaced by a Hamiltonian constraint $\mathcal{H} \approx 0$ together with momentum constraints $\mathcal{H}_i \approx 0$. The proper-time emergence extends to this setting through multi-constraint gauge fixing, time-dependent lapse and shift functions, and local proper time defined within each gauge patch. A programmatic approach to relaxing the static-background assumption, replacing global geometric symmetry with local, observer-accessible regularity conditions, is developed in Anonymous (2025a).

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